Interpreting Logical Expressions

Class 3
The syntactic road

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The syntactic road

Basic ideas :

- meaning as inferential role : what LC mean should be characterized in terms of inference rules,
- systematic analysis of meaning : the goal is to understand why we have the LC that we have,
- logical pluralism : there is a core meaning to LC acrosse different logics,
- the normativity of logic : the foes are not so much non-logical symbols than spurious logical constants which would render the system inconsistent.

Proof theoretic analysis of LC

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- natural deduction, with introduction and elimination rules,
- sequent calculus, with introduction rules on both sides of the turnstyle,
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structural analysis of LC within a sequent system :
K. Došen, 1989, "Logical Constants as Punctuation Marks", *NDJFL*.
G. Sambin, G. Battitlotti, Cl. Faggian, 2000, "Basic logic : reflection, symmetry, visibility", *JSL*.

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge$$

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Structural calculus : principles of ‘pure’ deduction which can be described independently of the constants of the object language.

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1. A structural language $\mathcal{L}$ whose vocabulary consists of :
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   - Two binary punctuation symbol : , and *vdash*
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For example :

$$\frac{\Gamma, A \vdash B \qquad \Delta \vdash B}{\Gamma, A, \Delta \vdash B}$$

should be read :

(((Γ and A) yields B) and (Δ yields B)) yields ((Γ and A and Δ) yields B)

Rules of $\mathcal{C}$ in Basic Logic

- Identity Axioms

  $A \vdash A$

- Exchange rules

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, B, A \vdash \Delta}\text{ Ex L} \qquad \frac{\Gamma \vdash A, B\Delta}{\Gamma \vdash B, A\Delta}\text{ Ex L}$$

- Cut

$$\frac{\Gamma \vdash A \qquad \Gamma' \vdash \Delta}{\Gamma'(\Gamma/A) \vdash \Delta}\text{ cut L} \qquad \frac{\Gamma \vdash \Delta' \qquad A \vdash \Delta}{\Gamma \vdash \Delta'(\Delta/A)}\text{ cut R}$$

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- But as in algebraic equations, the unknown element is only implicitly defined. The definitional equations? needs to be solved.
- Solving a Definitional Equation : given a couple of Sequent rules.

$$\frac{A, B \vdash \Delta}{A \otimes B \vdash \Delta} \; (\otimes\text{-Formation}) \quad \frac{A \otimes B \vdash \Delta}{A, B \vdash \Delta} \; (\otimes\text{-IR})$$

Find Explicit Reflection, the missing sequent rule equivalent to IR.

Solving Definitional Equations

1. Trivialization : Start from a $\otimes$-identity axiom, and apply IR to it :

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2. Composition of the axiom of $\otimes$-reflection with general premises :

$$\frac{\dfrac{\Gamma \vdash A \qquad A, B \vdash A \otimes B}{\Gamma, B \vdash A \otimes B} \qquad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B}$$

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3. We get the following equivalent rules :

$$\frac{A \otimes B \vdash \Gamma}{A, B \vdash \Gamma}\ (\otimes\text{-IR}) \quad \frac{\Gamma \vdash A \qquad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B}\ (\otimes\text{-ER})$$

Definitional equations

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \& B}\ (\&) \qquad \frac{A \vdash \Delta \qquad B \vdash \Delta}{A \oplus B \vdash \Delta}\ (\oplus)$$

$$\frac{A, B \vdash \Delta}{A \otimes B \vdash \Delta}\ (\otimes) \qquad \frac{\Gamma \vdash A, B}{\Gamma \vdash A \parr B}\ (\&)$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \bot}\ (\bot) \qquad \frac{\vdash \Delta}{1 \vdash \Delta}\ (1)$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}\ (\rightarrow) \qquad \frac{A \vdash B, \Delta}{A \leftarrow B \vdash \Delta}\ (\leftarrow)$$

Parameters

- Liberalizing contexts on the Left (L)

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\,(\otimes) \qquad \frac{\Gamma \vdash A \qquad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B}\,(\otimes\text{-ER})$$

- Liberalizing contexts on the Right (R)

$$\frac{A, B \vdash \Delta}{A \otimes B \vdash \Delta}\,(\otimes) \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma' \vdash \Delta' B}{\Gamma, \Gamma' \vdash \Delta, \Delta', A \otimes B}\,(\otimes\text{-ER})$$

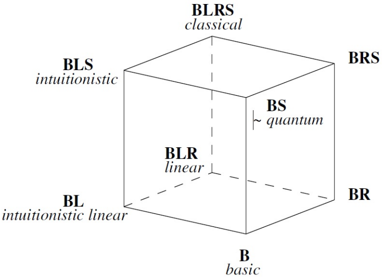
- Weakening (W)

$$\frac{\Gamma, \Gamma' \vdash \Delta}{\Gamma, \Sigma, \Gamma' \vdash \Delta}\,(wL) \qquad \frac{\Gamma \vdash \Delta, \Delta'}{\Gamma \vdash \Delta, \Sigma, \Delta'}\,(wR)$$

- Contraction (C)

$$\frac{\Gamma, \Sigma, \Sigma, \Gamma' \vdash \Delta}{\Gamma, \Sigma, \Gamma' \vdash \Delta}\,(cL) \qquad \frac{\Gamma \vdash \Delta, \Sigma, \Sigma, \Delta'}{\Gamma \vdash \Delta, \Sigma, \Delta'}\,(cR)$$

Unification in a Cube



from Sambin & alii, 2000

where $S = W + C$

Rules and analyticity

Why are the rules for a given LC valid ?
$\rightarrow$ because their validity is constitutive of the meaning of the LC.

conjunction is nothing but the logical concept which makes the rules for $\wedge$ valid.

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But isn't this too easy ? Not any kind of rules successfully define a LC !
A.N. Prior 1960, "The runabout inference-ticket", *Analysis*

$$\frac{A}{A \text{ Tonk } B} \qquad \frac{A \text{ Tonk } B}{B}$$

Definitional equations and the normativity of logic

Definitional equation for blonk (tonk lil' brother)

$$\frac{\Gamma \vdash A \qquad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \text{ blonk } B}\ (\text{blonk})$$

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Just like tonk, blonk makes the system inconsistent (take the axiom A blonk $B \vdash A$ blonk B and apply the rule upwards to get $\vdash B$).

Solving blonk's equation ?

$$\frac{\dfrac{A \text{ blonk } B \vdash A \text{ blonk } B}{A \text{ blonk } B \vdash A} \qquad A \vdash \Delta}{A \text{ blonk } B \vdash \Delta} \tag{1}$$

$\longrightarrow$ But then it's impossible to recover IR from ER.

$$\frac{\dfrac{A \text{ blonk } B \vdash A \text{ blonk } B}{\vdash B}}{A \text{ blonk } B \vdash B} \tag{2}$$

$\longrightarrow$ But then the proof is not done in $\mathcal{C}$.

Solvable Equations

Definition (solvable equation)

An equation is solvable iff there is an ER rule s.t.

- *one can derive ER from IR in $\mathcal{C}$*
- *one can derive IR from ER in $\mathcal{C}$*

Theorem

Solvability implies preservation of cut elimination

Concluding remarks

What is it exactly that makes LC logical ?

- the formality of the underlying structural calculus,
- the schematicity of the rules
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- uniqueness

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How does this relate to semantic criteria ?

- Alternative
- Supplement